Estimation and Identification of Nonlinear Dynamic Systems

D. Joseph Mook*
State University of New York at Buffalo, Buffalo, New York

A technique is presented for processing noisy state-observable time domain measurements of a nonlinear dynamic system in order to optimally estimate both the state vector trajectory and any model error that may be present. The model error estimate may be used subsequently to accurately identify the parameters in the differential equation model of the nonlinear dynamic system. The method is demonstrated by application to several examples and is shown to be accurate and robust with respect to 1) large errors in the original assumed differential equation model, including an assumed linear model, 2) low measurement frequency, 3) low measurement accuracy (i.e., large measurement noise), and even 4) low total number of measurements.

Introduction

THE widespread existence of nonlinear behavior in practical dynamic systems is well documented. Many excellent textbooks treat nonlinear subjects in great detail and include numerous references to practical systems of interest (e.g., Thompson and Stewart[1] or Nayfeh and Mook[2]). The current interest in large, flexible space structures, robotics, and other nonlinear technologies contributes many more examples of nonlinear behavior. Much of this work has been devoted to investigating the qualitative and quantitative behavior of various given nonlinear system models. Unfortunately, general methods for actually obtaining accurate differential equation models for nonlinear systems are not nearly as well developed as are the techniques for analyzing them.

Linear models continue to retain enormous popularity, no doubt due to the large volume of simple analytical results and tools available for their analysis. Moreover, most identification work to date has been restricted to linear models. Very often, nonlinear systems are assumed to be linearizable in some manner, and the resulting linear model then is used to analyze the behavior of the system. Perhaps the most common form of this approach is modal analysis, wherein the behavior of a dynamic system is represented by a finite linear combination of so-called system modes (e.g., Ewins[3,4]). Often, the linearized model is doomed to both quantitative and qualitative inadequacy. Quantitative inadequacy arises when the approximations and ignored terms required for the linearization produce inaccurate numerical results. More importantly, qualitative inadequacy arises when the linearized model is unable to predict certain qualitative aspects of the behavior, such as multiple steady states, drift, limit cycles, etc. Even systems that may be accurately modeled by linear models under certain conditions (small displacement, low stress and strain, etc.) normally become nonlinear if the conditions are violated.

If the dynamic model of the system is inaccurate, then a number of problems may arise. Analysis of the behavior of the system states may be unreliable. Inaccurate knowledge of the state behavior leads to difficulty in performance evaluation, control system design, and related tasks. Decisions regarding implementation, operation, and control of the system may become unreliable due to the uncertainty in the system.

A number of techniques for determining the parameters of nonlinear dynamic system models have been proposed. Ibáñez[5] used a describing function approach to estimate parameters in nonlinear dynamic systems. This approach assumes that the system response is dominated by a periodic response at the forcing frequency, and an approximate transfer function is constructed. Yun and Shinozuka[6] described an approach based on two versions of a Kalman filter, wherein the state vector is augmented by the addition of the unknown parameters. One version used an iterated linear filter-smoother algorithm, whereas the other version used an extended Kalman filter by assuming the nonlinear behavior to be representable by a linear departure motion about some known nominal state trajectory. Broersen,[7] using correlation techniques and statistical linearization, described a method for determining the coefficients in a linear combination of functions assumed to represent the system nonlinearity. This method requires measurements of all of the states and full knowledge of the system forcing, assumed to be random. Distefano and Rath[8] described three methods, including both direct and iterative techniques. More recently, Hanagud et al.[9] used the method of multiple scales to develop a technique for systems in which the nonlinearities are nominally small. A comparison between the methods of Refs. 8 and 9 also is contained in Ref. 9.

Identification work generally presupposes that a model form has been chosen and that the goal of the effort is to identify unknown parameters in the given model form. The unknown parameters then are determined by optimizing in some sense the fit of the chosen model form to the available data. Although the optimal values for the chosen model form may be determined, if the model form is incorrect, the identified model may be of low accuracy and usefulness. Recently, Hammond et al.[10] described a deconvolution technique for finding the form of unknown nonlinearities in vibrating systems.

The present work addresses these problems in two important ways. First, the method produces accurate state vector trajectory estimates even in the presence of incorrect model forms, so that analysis of the behavior of the system becomes much more reliable. For many problems of practical interest, accurate estimation of the unknown state trajectories is sufficient, e.g., in reconstructing the performance of the system during tests. The examples presented in a later section demonstrate that the present method may be used to obtain very accurate estimates of the system state trajectories, even in the presence of both large model error and large measurement error. The proposed method is much better suited for this task than Kalman filter algorithms, which assume that the model error is a zero-mean stochastic process of known covariance (e.g., Lewis[11]). Second, the examples presented in a later section demonstrate that the method is capable of accurately identifying unknown model parameters in nonlinear dynamic system models. The assumed model form contains the actual

Received Aug. 20, 1987; revision received April 21, 1988.

*Assistant Professor, Department of Mechanical and Aerospace Engineering.

model form but may include other terms as well. The identification procedure is shown to find zero coefficients for the incorrect terms in the assumed model form and near-perfect values for the correct term coefficients. The examples establish that the method is robust with respect to initial model ignorance, measurement accuracy, measurement frequency, and low total number of measurements.

Definition of the Problem

Consider a forced nonlinear dynamic system that may be accurately modeled by the equation

$$\ddot{x} + \omega_0^2 x = f[x(t),\dot{x}(t)] + F(t), \qquad t_0 \le t \le t_f \tag{1}$$

where x and $\dot{x}$ are the system states, $F(t)$ is a known external excitation, and $f[x(t),\dot{x}(t)]$ is a term that includes all of the system nonlinearities. The external excitation is assumed to be independent of the system states. Equation (1) may be converted to first-order state-space form as

$$\dot{z} = \begin{bmatrix} -0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} z + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ f[x(t),\dot{x}(t)] \end{Bmatrix} \tag{2}$$

where $z = \{x(t)\ \dot{x}(t)\}^T$.

Further assume that state-observable, discrete time domain measurements are available in the form

$$\tilde{y}(t_k) = g_k[z(t_k),t_k] + \nu_k, \quad t_0 \le t_k \le t_f, \quad k = 1,2,\ldots,M \tag{3}$$

where $\tilde{y}(t_k)$ is an $m \times 1$ measurement vector at time t_k, g_k is the accurate model of the measurement process, and ν_k represents measurement noise, assumed to be a zero-mean, Gaussian distributed process of known covariance R_k. The measurement noise assumption is consistent with the typical noise assumptions of optimal estimation theory (e.g., Lewis[11]).

In this paper, we address the very common situation in which the right-hand side of Eq. (2) is imperfectly known, specifically, that the function f is not known accurately, nor, perhaps, is the constant ω_0^2. The objective of this work may be stated as follows: given an imperfect state-space model of a nonlinear dynamic system, along with noisy state-observable measurements, obtain accurate estimates of 1) the unknown state trajectories $z(t)$ and 2) the unknown dynamic model error, i.e., the error on the right-hand side of Eq. (2). Further, if possible, 3) process the dynamic model error estimates in order to identify an accurate model of the dynamic system.

Formulation of the Solution Method

Suppose that a system is modeled by an equation in the form of Eq. (2), and further suppose that noisy, state-observable measurements in the form of Eq. (3) are available. The states calculated by integration of Eq. (2) may be substituted into the measurement model to predict the measurements. If the model is accurate, then the predicted measurements match the actual measurements with error residuals consistent with the measurement errors ν_k. This condition may be written in terms of the error covariances as

$$E(\{\tilde{y}(t_k) - g[z(t_k),t_k]\}\ \{\tilde{y}(t_k) - g[z(t_k),t_k]\}^T) = R_k \tag{4}$$

The quantity on the left-hand side of Eq. (4) is the expected value of the covariance of the residuals obtained by subtracting the predicted measurements from the actual measurements, and the quantity on the right-hand side is the actual measurement error covariance.

Now suppose that the model is not known accurately. Equation (4) may be interpreted as a constraint that must be satisfied by a model whose output is statistically consistent with the measured system output. Such a constraint is called a "covariance constraint" (Mook and Junkins[12]), and satisfaction of the covariance constraint may be used as a necessary condition for determining whether or not a candidate dynamic model is accurate. Specifically, let $\hat{z}(t)$ represent the state trajectories obtained by integrating a candidate model. If the covariance constraint described by

$$E(\{\tilde{y} - g[\hat{z}(t_k),t_k]\}\ \{\tilde{y}_k - g[\hat{z}(t_k),t_k]\}^T) \approx R_k \tag{5}$$

is not satisfied, then $\hat{z}(t)$ is not an accurate state vector estimate and the candidate model is not accurate. Note that the equality is assumed to be approximate because the measurement errors are random and only known statistically. Unless an infinite set of measurements is used, the measurement error covariance must be assumed to be approximate.

The formulation now proceeds as follows. Let a candidate (not necessarily accurate) model of the system be given by

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_0^2 & 0 \end{bmatrix} \hat{z} + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ f'(\hat{z}) \end{Bmatrix} \tag{6}$$

where the caret denotes estimated value, and the prime denotes the candidate form. Unless $f = f'$ and $\hat{\omega}_0^2 = \omega_0$, the estimates $\hat{z}(t)$ that result from integration of Eq. (6) do not satisfy the covariance constraint, Eq. (5). To account for such a model error, a model error term is added to Eq. (6) as

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_0^2 & 0 \end{bmatrix} \hat{z} + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ f'(\hat{z}) \end{Bmatrix} + \hat{d}(t) \tag{7}$$

where $\hat{d}(t)$ accounts for the model error. If $\hat{d}(t)$ is accurate, then the state estimates resulting from Eq. (7) satisfy the covariance constraint.

The optimality criterion used to determine $\hat{d}(t)$ is the minimization of the functional J, defined as

$$J = \sum_{k=1}^{M} \{\tilde{y}_k - g[\hat{z}(t_k),t_k]\}^T R_k^{-1} \{\tilde{y}_k - g[\hat{z}(t_k),t_k]\} + \int_{t_0}^{t_f} \hat{d}^T(\tau) W \hat{d}(\tau)\, \mathrm{d}\tau \tag{8}$$

Here, the first term is the weighted sum of the squares of the predicted-minus-actual measurement residuals, where the weighting is the inverse of the measurement error covariance R_k. Minimization of this term tends to force $\hat{z}(t_k)$ to predict the measurements exactly. However, if the assumed model is inaccurate, then $\hat{z}(t_k)$ cannot predict the measurements exactly, even if no measurement error is present, unless the model error estimate $\hat{d}(t)$ corrects the model. The second term in J penalizes the weighted integral square model error estimate $\hat{d}(t)$. Thus, the two components of J cannot be zero simultaneously if model error is present, since nonzero $\hat{d}(t)$ is necessary to correct the model. The appropriate balance between the two components of J is determined by satisfaction of the covariance constraint, Eq. (5), and is controlled by the weight W. Thus, W is used to enforce the covariance constraint, and minimization of J is used to determine $\hat{d}(t)$ for a given W.

The necessary conditions for the minimization of J with respect to $\hat{d}(t)$ form a two-point boundary-value problem (TPBVP) that may be summarized as (Mook and Junkins[12]):

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \hat{z}(t) + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ f'(\hat{z}) \end{Bmatrix} + \hat{d}(t) \tag{9}$$

$$\dot{\lambda} = \begin{bmatrix} 0 & \omega_0^2 \\ 1 & 0 \end{bmatrix} \lambda(t) \tag{10}$$

$$\hat{d}(t) = -\tfrac{1}{2} W^{-1} \lambda(t) \tag{11}$$

$$\hat{z}(t_0) = \text{specified, or } \lambda(t_0^-) = 0 \tag{12}$$

$$\lambda(t_k^+) = \lambda(t_k^-) + 2H_k R_k^{-1} \{\tilde{y}_k - g[\hat{z}(t_k),t_k]\} \tag{13}$$

$$H_k = \frac{\partial g}{\partial z}\bigg|_{\hat{z}(t_k),t_k}$$

$$\hat{z}(t_f) = \text{specified, or } \lambda(t_f^+) = 0 \tag{14}$$

Here, λ is a vector of Lagrange multipliers often called "costates," with dimension equal to that of $\hat{z}$. Since half of the boundary conditions are known at t_0 and the other half at t_f, Eqs. (9–14) constitute a TPBVP. Numerous excellent methods for solving TPBVP's have been described in the literature (e.g., Keller[13]). A closed-form solution based on multiple shooting has recently been developed for this class of TPBVP's (Lew and Mook[14]), when the differential equations, Eqs. (9) and (10), are linear.

Solution of the TPBVP yields the state estimates $\hat{z}(t)$ and the model error estimates $\hat{d}(t)$. The state estimates next are substituted into the measurement model in order to determine if the covariance constraint has been satisfied. If so, the state estimates are statistically consistent with the measurements. If the covariance constraint has not been satisfied, a new value for W is chosen and the resulting TPBVP solution is obtained.

The choice of a new value for W depends on the manner in which the covariance constraint is violated. If the predicted measurements match the actual measurements too closely, then the left-hand side of Eq. (5) is smaller than the right-hand side. The model has been "over-corrected" [i.e., too much $\hat{d}(t)$], and so the value of W should be increased to more heavily penalize $\hat{d}(t)$ in J. On the other hand, if the predicted measurements do not match the actual measurements closely enough, then too little $\hat{d}(t)$ has been added, and the value of W should be decreased. When the two sides of Eq. (5) are approximately equal, the covariance constraint is satisfied.

The determination of an appropriate value of W may be performed using either analytical methods or trial and error. The author has used typical optimization approaches, such as the gradient method, Gauss-Newton, etc., to derive analytical searches for W that minimize an error function of the residuals between the left- and right-hand sides of Eq. (5). Alternatively, the W search may be performed via trial and error by simply increasing or decreasing W, respectively, if the left-hand side of Eq. (5) is less than or greater than the right-hand side. Each TPBVP solution is typically obtained in a few seconds on a VAX 780, so that a trial and error search for W may be performed interactively while working at a terminal. In either case, the entire solution for the examples presented later usually is obtained in 5 min or less of clock time (perhaps 1 min of CPU time).

Inspection of Eq. (13) reveals that the costates may contain a jump discontinuity at a measurement time. Consequently, due to Eq. (11), the $\hat{d}(t)$ estimates also may contain a jump discontinuity at the measurement times. The size of the jump is linearly proportional to the residual between the predicted measurements and the actual measurements. When the covariance constraint is satisfied, the size of this residual is approximately equal to the noise in the measurements. Thus, if the measurements are poor, the model error estimate may contain relatively large jump discontinuities at the measurement times. If the measurements are accurate, the model error estimate is essentially smooth. In all cases, the state trajectory estimates are continuous. Note in passing that state estimates obtained using Kalman filters or filter-like algorithms contain state jump discontinuities at the measurement times (e.g., Gelb[15]). The existence of jumps in the model error estimates does not affect their use in determining the unknown model coefficients, as explained in the next section.

Parameter Identification Via Postprocessing of Model Error Estimates

The solution of the TPBVP yields estimates for $\hat{z}(t)$ and $\hat{d}(t)$, which, if the covariance constraint is satisfied, are optimal. The model error estimate $\hat{d}(t)$ is accurate and may be used to construct the unknown terms on the right-hand side of Eq. (2). In general, this is an artistic undertaking. However, if the model error is itself modeled using typical nonlinear terms with unknown coefficients, the coefficients may be optimally estimated by using least squares to fit the model error model to the model error estimate. Some examples of this procedure are given here for illustration.

Suppose that a given nonlinear oscillator may be accurately modeled by the equation

$$\ddot{x} + \omega_0^2 x + \alpha x^2 + \beta x^3 + \delta \dot{x} = F(t) \tag{15}$$

but that the values of α, β, and δ are unknown. Following the procedure of the preceding section, state-observable measurements of $z = \{x\ \dot{x}\}^T$ are used to determine estimates of $\hat{d}(t)$ and $\hat{z}(t)$. Without the unknown terms in the candidate model used in the present estimation procedure, the model has the form

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_0^2 & 0 \end{bmatrix} \hat{z}(t) + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \hat{d}(t) \tag{16}$$

Comparing Eq. (15) with Eq. (16), the model error is clearly modeled by

$$\hat{d}_2(t) = \alpha \hat{z}_1^2(t) + \beta \hat{z}_1^3(t) + \delta \hat{z}_2(t) \tag{17}$$

Equation (17) represents a single equation in the three unknown constants α, β, and δ. However, estimates of $\hat{d}(t)$ are available for $t_0 \le t \le t_f$. Consequently, Eq. (17) may be sampled repeatedly to obtain

$$\begin{Bmatrix} d_2(t_1) \\ d_2(t_2) \\ \cdot \\ \cdot \\ \cdot \\ d_2(t_\ell) \end{Bmatrix} = \begin{bmatrix} z_1^2(t_1) & z_1^3(t_1) & z_2(t_1) \\ z_1^2(t_2) & z_1^3(t_2) & z_2(t_2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ z_1^2(t_\ell) & z_1^3(t_\ell) & z_2(t_\ell) \end{bmatrix}$$

$$\times \begin{Bmatrix} \alpha \\ \beta \\ \delta \end{Bmatrix}, \quad t_0 \le t_1 < t_2 < \ldots < t_\ell \le t_f \tag{18}$$

or

$$D_{\ell \times 1} = A_{\ell \times 3} P_{3 \times 1}$$

where $\ell > 3$. Each d_i and z_i in Eq. (18) is an estimate, but the carets have been omitted due to space. Generally, it is desirable to choose large ℓ so that the system of equations in Eq. (18) is highly overdetermined. Then, the least-squares estimate of the unknown constants, obtained by minimizing

$$\phi = [D - A\hat{P}]^T [D - A\hat{P}] \tag{19}$$

with respect to $\hat{P}$, is

$$\hat{P} = (A^T A)^{-1} A^T D \tag{20}$$

Note that the inverse in Eq. (20) is only 3×3 (for 3 unknown constants) regardless of the size of ℓ. The inverse in Eq. (20) need not be calculated if the equation is premultiplied by $A^T A$ and solved using a linear equation algorithm for the three simultaneous equations.

The procedure described by Eqs. (15–20) may be readily applied to find the best least-squares fit of any assumed model. If no a priori knowledge of the model other than the external forcing is assumed, the candidate model appears as

$$\dot{\hat{z}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{z} + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} + \hat{d}(t) \tag{21}$$

and the entire homogeneous model is contained in the model error term $\hat{d}(t)$. The four unknowns ω_0^2, α, β, and δ are estimated using the procedure of Eqs. (15-20), modified in the obvious way. The unknown parameter vector P is now of length 4, and an additional column consisting of $\hat{z}_1(t_1)$, $\hat{z}_1(t_2)$, $\ldots$, $\hat{z}(t_\ell)$ is added to the A matrix of Eqs. (18-20). The necessary matrix inverse in Eq. (20) is in this case 4×4 (or, there are four linear equations to be solved simultaneously). One of the examples illustrates this case.

As previously discussed, the model error estimate $\hat{d}(t)$ may contain jump discontinuities at the measurement times if the measurements are very noisy. However, this situation does not affect the least squares identification procedure, provided that the times chosen to construct Eq. (18) do not coincide with the measurement times. Furthermore, even if the measurements are very noisy, the jumps do not occur in $d_2(t)$ unless the measurement model g_k explicitly includes terms in $\dot{x}(t)$ [see Eq. (13), and the definition of H_k]. For example, in a mechanical system, if the measurements are functions of position $x(t)$ only, then regardless of the measurement noise, no jumps are introduced into $d_2(t)$.

Examples

In this section, brief formulations for some examples are presented in addition to extensive results that demonstrate the accuracy and robustness of the technique.

Assumed Models

An infinity of erroneous models may be chosen for a given nonlinear dynamic system. In this paper, two general incorrect model forms are used for illustration purposes: 1) a linear oscillator model and 2) a "total ignorance" model.

The linear oscillator has the form

$$\ddot{x} + \delta\dot{x} + \omega_0^2 x = F(t)$$

which may be written as a linear state-space model in the form

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\delta \end{bmatrix} z(t) + \begin{Bmatrix} 0 \\ F(t) \end{Bmatrix} \tag{22}$$

The behavior of linear oscillators is very well known and documented, and consequently many vibrating systems are modeled using them. A practical test of the present method is now investigated by estimation and identification of a nonlinear oscillator, using a linear oscillator as the assumed model. This situation arises in practice whenever a linear model of a nonlinear oscillator is assumed.

The following equation, adopted from Ref. 1, is used as the true nonlinear oscillator:

$$2.56\ddot{u} + 0.32\dot{u} + u + 0.05u^3 = 2.5\cos t \tag{23}$$

This example is chosen because it exhibits multiple steady-state solutions, depending on the initial conditions. This phenomenon does not exist for linear oscillators. Simulated measurements are created by adding a computer-generated, Gaussian-distributed random number to the true solution of Eq. (23) at discrete times as

$$\tilde{y}(t_k) = u(t_k) + v_k \tag{24}$$

where $E\{v_k\} = 0$ and $E\{v_k v_k^T\} = R_k$. The value of R_k may be chosen to simulate high or low accuracy when the measurements are created. The assumed (but erroneous) linear oscillator model is given by

$$2.56\ddot{\hat{u}} + 0.32\dot{\hat{u}} + \hat{u} = 2.5\cos t \tag{25}$$

where the cubic nonlinear term is ignored. The method of the paper now is used to 1) obtain accurate $\hat{u}(t)$ and $\dot{\hat{u}}(t)$, 2) obtain an accurate model error estimate $\hat{d}(t)$, and 3) process the model error estimate to obtain the missing nonlinear term.

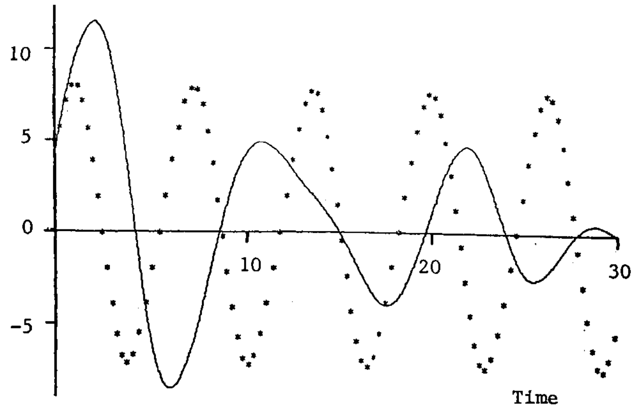

Fig. 1 Assumed linear model and 100 noiseless measurements (*).

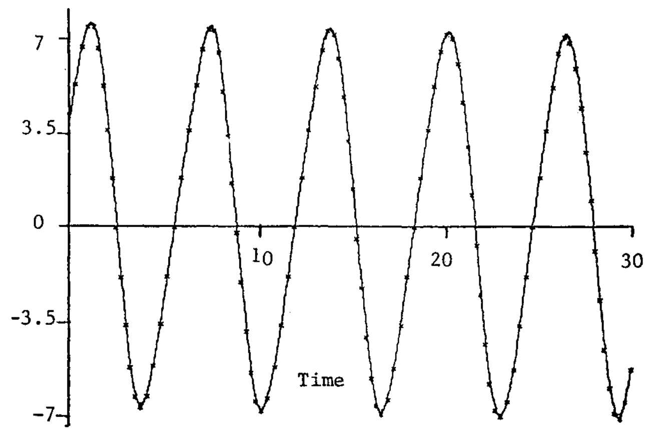

Fig. 2 True, measured, and estimated position.

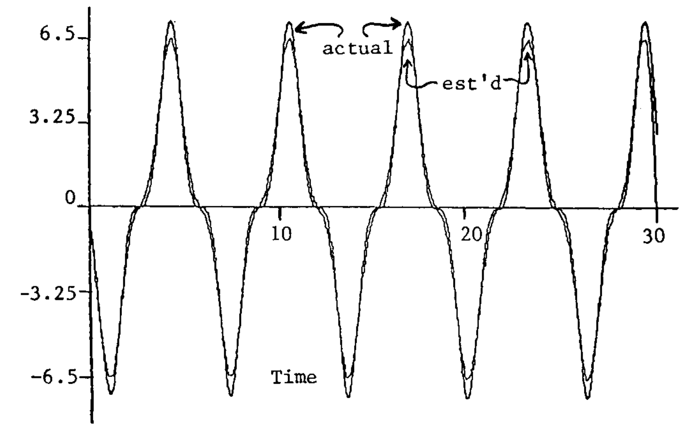

Fig. 3 Actual and estimated model error.

In Fig. 1, the uncorrected model output and a set of simulated measurements are shown for initial conditions of $u(o) = 3.15$ and $\dot{u}(o) = 5.81$. The measurements span a time domain of 0-30, representing slightly less than five cycles of oscillation, and contain no noise. The error variance of the measurements is 0, while the error variance of the model is 6.57. Note that the assumed linear model, which ignores the nonlinear term $0.05u^3$, is very poor.

In Fig. 2, the estimate obtained by processing the measurements with the present method is plotted along with the truth. Clearly, the state estimate is very accurate, with an error variance of 0.045. Not pictured for conciseness, but also very accurate, is the estimate of $\dot{\hat{u}}$. Thus, the accurate state estimate objective is clearly satisfied.

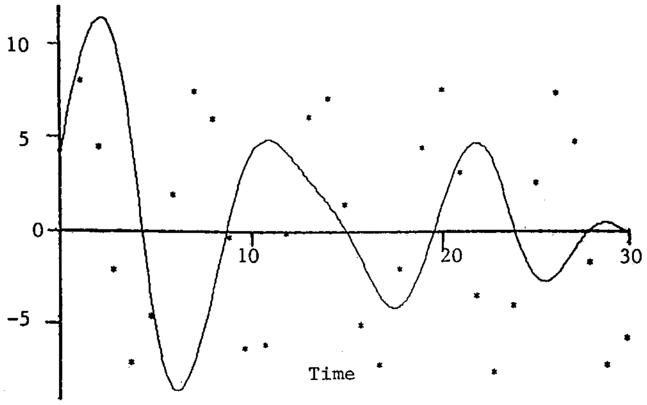

Fig. 4 Assumed linear model and 30 noiseless measurements (*).

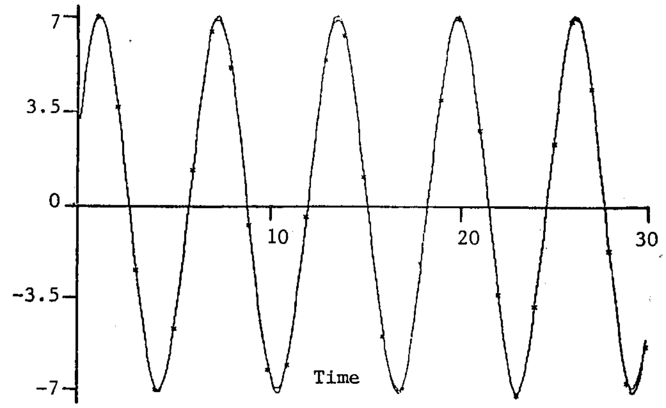

Fig. 5 True, measured, and estimated position.

In Fig. 3, the true model error $[0.05u^3(t)]$ is plotted along with the estimated model error $\hat{d}(t)$, which is seen to be very accurate. The estimate slightly misses the peaks of the true model error. This is due to the second term in J [in Eq. (8)] which, when minimized, tends to keep $\hat{d}(t)$ as small as possible. Nevertheless, an obviously accurate state estimate is obtained even though the model correction estimate contains slight errors at the extreme values. The accurate model error estimation objective is clearly satisfied.

The results pictured in Figs. 1-3 are often sufficient for many practical applications. However, the model error estimate in Fig. 3 may be further processed via least squares to determine unknown model terms, as outlined in the previous section. Here, we assume a model of the model error as

$$\hat{d}(t) = \alpha u^2(t) + \beta u^3(t) \tag{26}$$

[see Eq. (17)], which contains commonly used nonlinear terms, the quadratic and the cubic. Using the method of Eqs. (15-20), estimates of α and β are obtained as

$$\alpha \approx 10^{-5}, \qquad \beta \approx 0.0492 \tag{27}$$

which are both very accurate (the values are 0 and 0.05, respectively). The identification of the nonlinear model is very accurate.

The results of this first test case represent measurement frequency of approximately 20 measurements/cycle, perfect measurement accuracy, and 100 total measurements of a single output. Currently available instrumentation is capable of a/d conversion in excess of 1 mHz, so that a measurement frequency of 20/cycle or greater is possible for systems vibrating at up to 50 kHz and above. Obtaining 100 measurement samples in such a situation is clearly trivial. The measurement accuracy will depend on the sensor, but near-perfect accuracy is now possible in many practical systems. Consequently, the highly accurate results obtained in this first case are indicative of results obtainable for many practical systems.

Nevertheless, we now proceed to investigate other cases in which the model and/or the measurements are inferior to the first case. In Fig. 4, the same model is plotted along with only 30 perfect measurements. Figure 5 shows the estimate, the truth, and the measurements. Again, the state estimate accuracy is excellent, with error variance equal to 0.092. The measurement frequency in this case is approximately 6 measurements/cycle. The model error estimate is plotted in Fig. 6 along with the true model error. Although the measurement frequency is insufficient to fully capture every detail, especially the peaks and the "shoulders" near 0, the model error estimate closely tracks the true model error. A least-squares fit of this model error estimate using an assumed error model of $\hat{d}(t) = \alpha\hat{u}^2(t) + \beta\hat{u}^3(t)$ produces $\alpha \cong 10^{-5}$, $\beta = 0.0483$; again, parameter recovery is excellent.

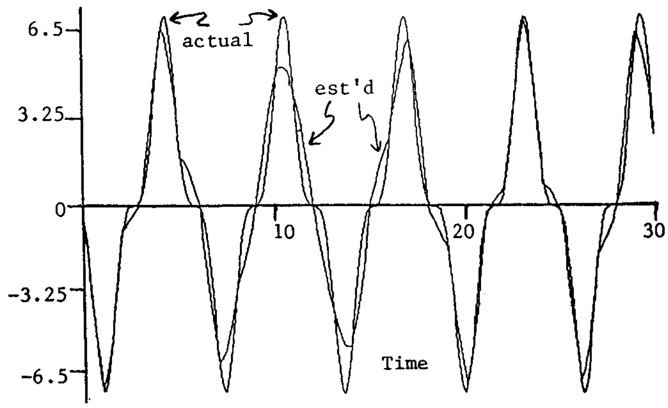

Fig. 6 Actual and estimated model error.

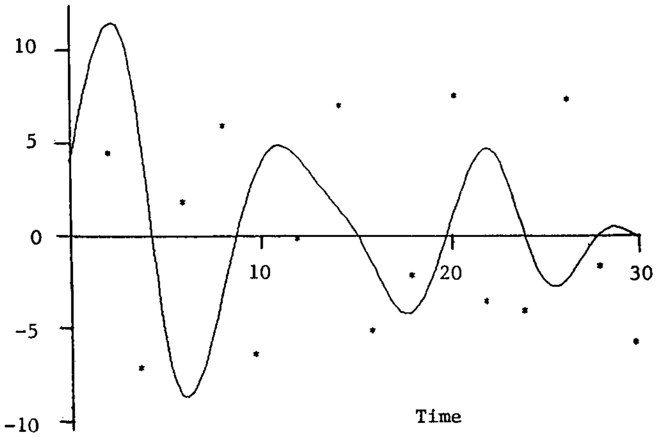

Fig. 7 Assumed linear model and 15 noiseless measurements (*).

In Figs. 7-9, results are shown for the same problem, except that only 15 measurements are used. Figure 7 shows the assumed model and the measurements, Fig. 8 shows the estimate, truth, and measurements, and Fig. 9 shows the model error estimate and the true model error. In Fig. 8, a small estimate error is apparent near the unmeasured peaks. However, considering how poor the model is and how scarce the measurements are, the state estimate is remarkably accurate. The estimate error variance is 0.295. In Fig. 9, the estimated model error is seen to miss the peaks and the shoulders. However, a least-squares fit using a model error model of $d(t) = \alpha\hat{u}^2(t) + \beta\hat{u}^3(t)$ produces $\alpha \cong 10^{-5}$ and $\beta = 0.046$. The measurement frequency in this case is only approximately 3 measurements/cycle.

We now proceed to consider the effects of measurement noise. Figures 10-12 show results obtained using 100 noisy

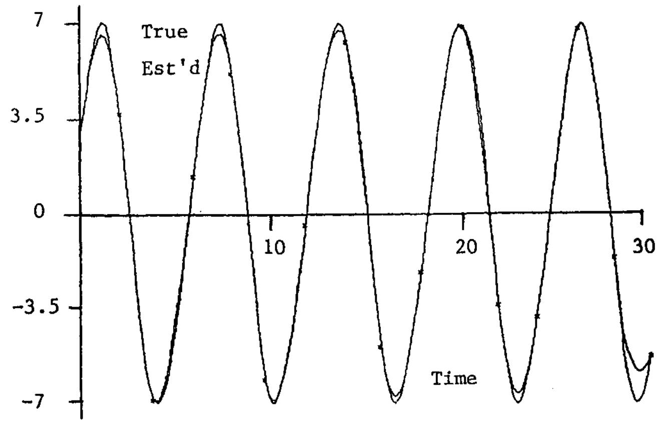

Fig. 8 True, measured, and estimated position.

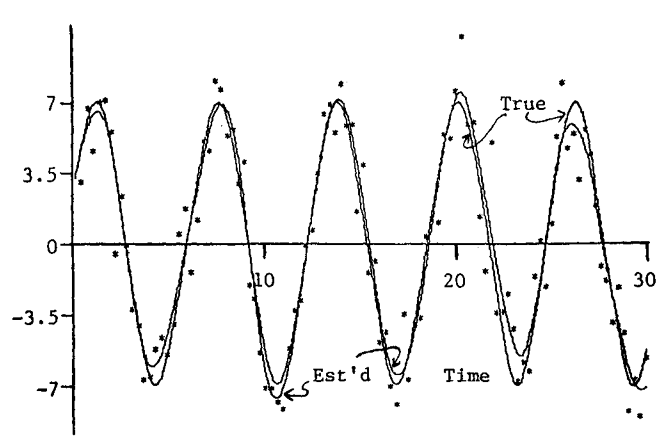

Fig. 11 True, measured, and estimated position.

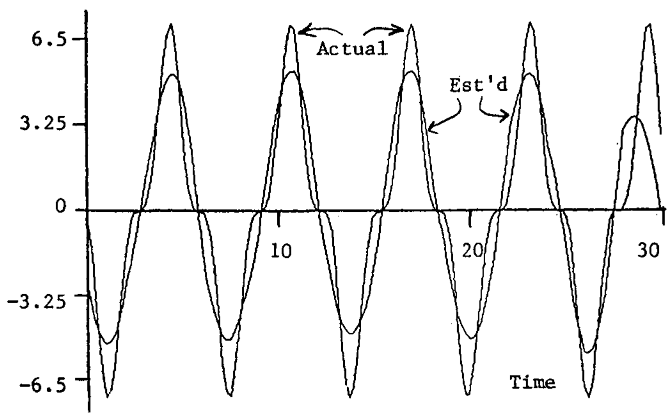

Fig. 9 Actual and estimated model error.

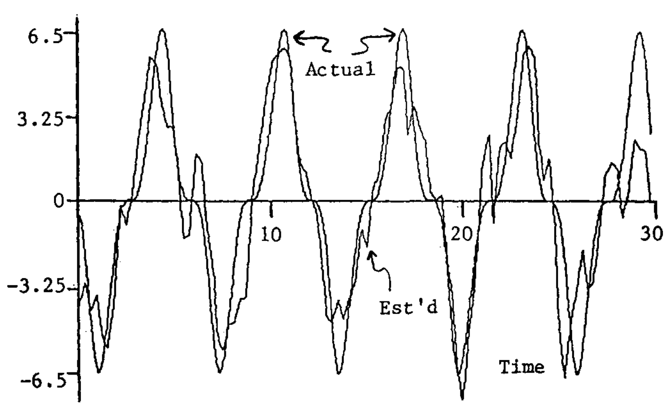

Fig. 12 Actual and estimated model error.

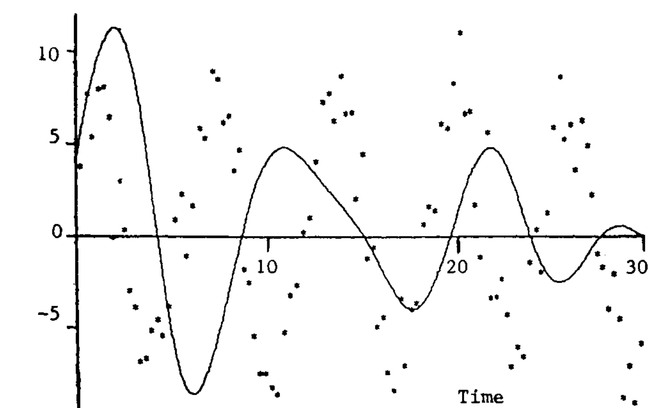

Fig. 10 Assumed linear model and 100 noisy measurements (*). Measurement noise approximately 30%.

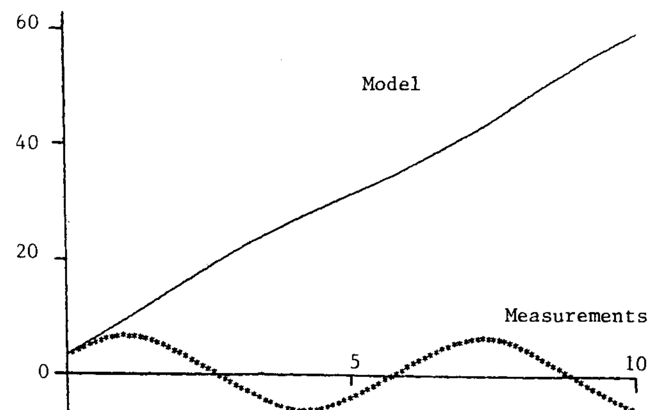

Fig. 13 Assumed "total ignorance" model and noiseless measurements (*).

measurements spanning $0 \leq t \leq 30$. The measurement error variance is 1.63, representing a very noisy measurement error of approximately 30% of the average signal. This level of error is much greater than that typically found in practice. The assumed model, with error variance 6.57, and the measurements, with error variance 1.63, are plotted in Fig. 10. In Fig. 11, the estimate $\hat{u}$ is plotted along with the truth and the measurements. The estimate error variance is 0.34. Clearly, the state estimate is much more accurate than either the model or the measurements. In Fig. 12, the model error estimate and true model error are plotted. Due to the noise in the measurements, the model error estimate is very noisy. Nevertheless, a least-squares fit of this error estimate using an error model of $d(t) = \alpha\hat{u}^2(t) + \beta\hat{u}^3(t)$ produces estimates $\alpha \cong 10^{-4}$ and $\beta = 0.042$. Although these parameter estimates are not as accurate as those obtained using perfect measurements, they are still quite accurate.

We now proceed to consider an even poorer assumed model. In Figs. 13–15, the assumed model is

$$2.56\ddot{u} = 2.5 \cos t \tag{28}$$

Comparing Eq. (28) with Eq. (23), all of the model, except for the known external forcing, has been ignored, representing "total ignorance" of the oscillator.

In Fig. 13, the (unstable) model is plotted along with 100 perfect measurements spanning $0 \leq t \leq 10$. The model is very poor and has an error variance of 37.2. In Fig. 14, the estimate is plotted along with the truth and the measurements. The estimate is accurate and has an error variance of 0.0015.

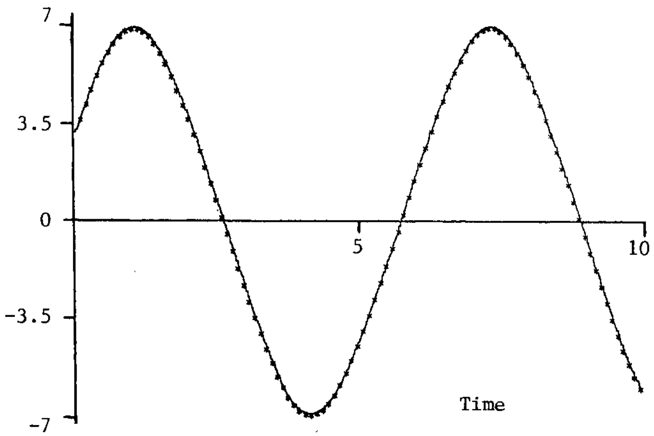

Fig. 14 True, measured, and estimated position.

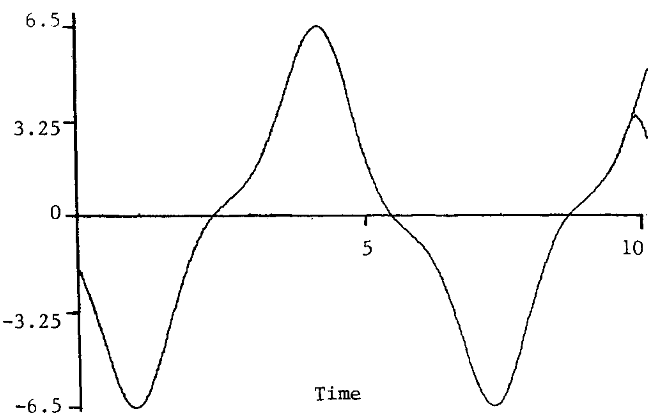

Fig. 15 Actual and estimated model error.

In Fig. 15, the model error estimate is plotted along with the true model error. Except for the region around $t = 10$, the model error estimate is excellent. The final error is due to the assumed boundary condition at $t = 10$, namely, that the state is unknown, which requires the costate (and hence $\hat{d}$) to go to zero.

The parameter estimation was performed using an error model of $\hat{d}(t) = \alpha\hat{u}^2(t) + \beta\hat{u}^3(t) + \delta\dot{\hat{u}}(t) + \omega_0^2\hat{u}(t)$ and resulted in estimates of $\alpha \cong 10^{-5}$, $\beta = 0.0495$, $\delta = 0.319$, and $\omega_0^2 = 1.01$, compared with the true values of 0, 0.05, 0.32, and 1, respectively.

Summary and Conclusions

In this paper, a novel method for accurately estimating and identifying nonlinear dynamic systems is derived and demonstrated on a number of example problems. The results of the examples show that the method produces accurate state trajectory estimates and accurate model error estimates under a wide range of conditions. The method is very robust with respect to the accuracy of the assumed model (model accuracy as low as "total ignorance"), the accuracy of the state measurements (measurement error as high as 30%), the frequency of the state measurements (measurement frequency as low as 3 measurements/cycle), and very low total number of measurements (total number of measurements as low as 15). The technique is based on optimally determining the state and model error estimates through satisfaction of a covariance constraint.

The algorithm is easy to implement and computationally efficient. The state and model error estimations do not require a correctly assumed model form. The parameter estimation is based on a deterministic least-squares formulation and does not require iteration, as do many existing techniques. The required measurements need be only state-observable, discrete time measurements, and measurement noise is specifically accounted for by the covariance constraint.

References

[1] Thompson, J. M. T. and Stewart, H. B., *Nonlinear Dynamics and Chaos,* Wiley, New York, 1986.

[2] Nayfeh, A. H. and Mook, D. T., *Nonlinear Oscillations,* Wiley, New York, 1979.

[3] Ewins, D. J., *Modal Testing: Theory and Practice,* Research Studies Press, Ltd., Wiley, New York, 1985.

[4] *Proceedings of the International Modal Analysis Conference (IMAC),* Union College, Schenectady, NY, 1980, 1982, 1984, 1986.

[5] Ibáñez, P., "Identification of Dynamic Parameters of Linear and Nonlinear Structural Models from Experimental Data," *Nuclear Engineering and Design,* Vol. 25, June 1973, pp. 30–41.

[6] Yun, C.-B. and Shinozuka, M., "Identification of Nonlinear Structural Dynamic Systems," *Journal of Structural Mechanics,* Vol. 8, April 1980, pp. 187–203.

[7] Broerson, P. M. T., "Estimation of Parameters of Nonlinear Dynamical Systems," *International Journal of Nonlinear Mechanics,* Vol. 9, Oct. 1974, pp. 355–361.

[8] Distefano, N. and Raith, A., "System Identification in Nonlinear Structural Seismic Dynamics," *Computer Methods in Applied Mechanics and Engineering,* Vol. 5, May 1975, pp. 353–372.

[9] Hanagud, S. V., Meyyappa, M., and Craig, J. I., "Method of Multiple Scales and Identification of Nonlinear Structural Dynamic Systems," *AIAA Journal,* Vol. 23, May 1985, pp. 802–807.

[10] Hammond, J. K., Lo, H. R., and Smith, J. S., "Identification of Nonlinearities in Vibrating Systems Using Optimal Control Techniques," *Proceedings of the 5th International Modal Analysis Conference,* London, April 1987, Union College, Schenectady, NY, pp. 1467–1473.

[11] Lewis, F. L., *Optimal Estimation,* Wiley, New York, 1986.

[12] Mook, D. J. and Junkins, J. L., "Minimum Model Error Estimation of Poorly Modeled Dynamic Systems," AIAA Paper 87-0173, Jan. 1987.

[13] Keller, H. B., *Numerical Solution of Two Point Boundary Value Problems,* SIAM, No. 24, Regional Conference Series in Applied Mathematics, 1976.

[14] Lew, J.-H. and Mook, D. J., "Multiple Shooting for Linear Two-Point Boundary Value Problems with Jump Discontinuities" *SIAM Journal of Numerical Analysis* (to be published).

[15] *Applied Optimal Estimation,* edited by A. Gelb, M.I.T. Press, Cambridge, MA, 1974.